# A note on spontaneous symmetry breaking in quantum field theory

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## Preparation for SSB

### Noether's theorem

Since the Lagrangian is invariant under the infinitismall action

$$0 = \delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\varphi_a}\delta\varphi_a + \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_a}\delta\partial_\mu\varphi_a.$$

and using the equation of motion

$$\frac{\delta\mathcal{L}}{\delta\varphi_a} = \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_a}\right)$$

then we have

$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_a}\delta\varphi_a\right) = 0$$

defining the current $j^\mu$ as

$$J^\mu \equiv \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_a}\delta\varphi_a$$

then it always preserves

$$\partial_\mu J^\mu = 0$$

### 0.1 Infinitismall action

Denote $\vec{varphi}$ to be a vector and consider the transformation

$$\vec{\varphi} \to e^{\theta^A T^A}\vec{\varphi}$$

when all $\theta^A$ are very small

$$\begin{aligned} &\sim \left(1 + \theta^A T^A\right)\vec{\varphi} \\ &= \vec{\varphi} + \theta^A\left(T^A\right)_{ab}\vec{\varphi}_b \end{aligned}$$

we can consider $\delta\varphi$ as

$$\delta\vec{\varphi}_a = \theta^A\left(T^A\right)_{ab}\vec{\varphi}_b$$

using this, the current can be shown as

$$J^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_a}\theta^A\left(T^A\right)_{ab}\varphi_b$$

This $T^A$ are the generators of Lie algebra. When $e^{\theta^A T^A}$ is orthogonal matrix, then the $T^A$ are antisymmetric matrices, and when $e^{\theta^A T^A}$ is unitary matrix, then the $T^A$ are Hermite matrices. I am going to take an instance

$$\varphi = \frac{1}{2}\left[(\partial\vec{\varphi})^2 - m^2\vec{\varphi}^2\right] - \frac{\lambda}{4}\left(\vec{\varphi}^2\right)^2$$

and denote $\vec{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$, then this Lagrangian is invariant under the action of $O(3)$, and the generator of $\mathfrak{o}(3)$ is

$$\mathfrak{o}(3) = \left\langle \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \right\rangle$$

so

$$\begin{aligned} \delta\varphi_a &= \theta^A \left(T^A\right)_{ab} \varphi_b \\ &= i \begin{pmatrix} 0 & -\theta^1 & \theta^2 \\ \theta^1 & 0 & -\theta^3 \\ -\theta^2 & \theta^3 & 0 \end{pmatrix} \varphi \end{aligned}$$

Note that you can take the generator as you want, but they are one of the way to take.

$$= i\left(-\theta^1\varphi^2 + \theta^2\varphi^3, \quad \theta^1\varphi^1 - \theta^3\varphi^3, -\theta^2\varphi^1 + \theta^3\varphi^2\right)$$

now taking $\theta^i = 1$, and

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_1} &= \partial^\mu\varphi_1 \\ \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_2} &= \partial^\mu\varphi_2 \\ \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_3} &= \partial^\mu\varphi_3 \end{aligned}$$

Thus

$$J^\mu = i\left[\partial^\mu\varphi_1\left(-\varphi_2 + \varphi_3\right) + \partial^\mu\varphi_2\left(\varphi_1 - \varphi_3\right) + \partial^\mu\varphi_3\left(-\varphi_1 + \varphi_3\right)\right]$$

especially

$$J^0 = i\left[\dot{\varphi}_1\left(-\varphi_2 + \varphi_3\right) + \dot{\varphi}_2\left(\varphi_1 - \varphi_3\right) + \dot{\varphi}_3\left(-\varphi_1 + \varphi_3\right)\right]$$

# 1 SSB of complex scalar field

we recall the complex scaler field

$$\mathcal{L} = \partial\varphi^\dagger\partial\varphi - m^2\varphi^\dagger\varphi - \lambda\left(\varphi^\dagger\varphi\right)^2$$

letting $m \to i\mu$ (then this term is not mass term any longer)

$$\mathcal{L}' = \partial\varphi^\dagger\partial\varphi + \mu^2\varphi^\dagger\varphi - \lambda\left(\varphi^\dagger\varphi\right)^2$$

since the field $\varphi$ can be written as

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$$

substituting

$$\begin{aligned} \mathcal{L}' = \frac{1}{2}\left(\partial\varphi_1 - i\partial\varphi_2\right)\left(\partial\varphi_1 + i\partial\varphi_2\right) + \frac{1}{2}\mu^2\left(\varphi_1 - i\varphi_2\right)\left(\varphi_1 + i\varphi_2\right) \\ - \frac{1}{4}\lambda\left\{\left(\varphi_1 - i\varphi_2\right)\left(\varphi_1 + i\varphi_2\right)\right\}^2 \\ = \frac{1}{2}\left(\partial\varphi_1\right)^2 + \frac{1}{2}\left(\partial\varphi_2\right)^2 + \frac{1}{2}\mu^2\left(\varphi_1^2 + \varphi_2^2\right) \\ - \frac{\lambda}{4}\left(\varphi_1^2 + \varphi_2^2\right)^2 \end{aligned}$$

you will soon realize the above is

$$= \frac{1}{2}(\partial\vec{\varphi})^2 + \frac{\mu^2}{2}\vec{\varphi}^2 - \frac{1}{4}\lambda\left(\vec{\varphi}^2\right)^2$$

so the potential is

$$V(\vec{\varphi}^2) = -\frac{\mu^2}{2}\vec{\varphi}^2 + \frac{\lambda}{4}\left(\vec{\varphi}^2\right)^2$$

and this is quadratic equation of $\vec{\varphi}^2$, solving This

$$\frac{dV}{d\vec{\varphi}^2} = -\frac{1}{2}\mu^2 + \frac{1}{2}\lambda\vec{\varphi}^2 = 0$$

$$\Rightarrow \vec{\varphi}^2 = \frac{\mu^2}{\lambda}$$

in other words the circle is minima of the potential $V$ of this Lagrangian

$$\varphi_1^2 + \varphi_2^2 = \frac{\mu^2}{\lambda}$$

now we take

$$(\varphi_1, \varphi_2) \to \left(\varphi_1' + \frac{\mu}{\sqrt{\lambda}}, \varphi_2'\right)$$

substituting $\mathcal{L}$ assuming $v = \frac{\mu}{\sqrt{\lambda}}$

$$\begin{aligned}
\mathcal{L} \to & \frac{1}{2}\left(\partial\varphi_1'\right)^2 + \frac{1}{2}\left(\partial\varphi_2'\right)^2 + \frac{1}{2}\mu^2\left\{\left(\varphi_1' + v\right)^2 + \varphi_2'^2\right\} \\
& - \frac{\lambda}{4}\left\{\left(\varphi_1' + v\right)^2 + \varphi_2'^2\right\}^2 \\
= & \frac{1}{2}\left(\partial\varphi_1'\right)^2 + \frac{1}{2}\left(\partial\varphi_2'\right)^2 + \frac{\mu^2}{2}\left(\varphi_1'^2 + 2\varphi_1' v + v^2 + \varphi_2'^2\right) \\
& - \frac{\lambda}{4}\left(\varphi_1'^2 + 2\varphi_1' v + v^2 + \varphi_2'^2\right)^2
\end{aligned}$$

calculating straightforward

$$\mathcal{L} = -\frac{1}{4}\lambda\varphi_1^4 - \frac{1}{2}\lambda\varphi_1^2\varphi_2^2 - \frac{1}{4}\lambda\varphi_2^4 - \sqrt{\lambda}\mu\varphi_1^3 - \sqrt{\lambda}\mu\varphi_1\varphi_2^2 - \mu^2\varphi_1^2 + \frac{\mu^4}{4\lambda} + \frac{1}{2}\partial\varphi_1^2 + \frac{1}{2}\partial\varphi_2^2$$

Make this simpler.

$$= \frac{1}{2}(\partial\vec{\varphi}')^2 - \mu^2{\varphi_1'}^2 + \frac{\mu^4}{4\lambda} + \mathcal{O}$$

this means $\varphi_1'$ is massive field, but $\varphi_2'$ is no longer massive, but a massless field.

## 1.1 Consideration of the above

Denote $m = \sigma + i\tau \quad (\sigma_1\tau \in \mathbb{R})$, we are going to consider the Lagrangian

$$\mathcal{L} = \partial\varphi^\dagger\partial\varphi - m^2\varphi^\dagger\varphi - \lambda\left(\varphi^\dagger\varphi\right)^2$$

caliculating

$$\begin{aligned}
\mathcal{L} &= \left(\partial\varphi^\dagger\right)\left(\partial\varphi\right) - \left(\sigma^2 - \tau^2 + 2i\sigma\tau\right)\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2 \\
&= \left(\partial\varphi^\dagger\right)\left(\partial\varphi\right) - \left(\sigma^2 - \tau^2\right)\varphi^\dagger\varphi - \lambda\left(\varphi^\dagger\varphi\right)^2 \\
&\quad - 2i\sigma\tau\varphi^\dagger\varphi
\end{aligned}$$

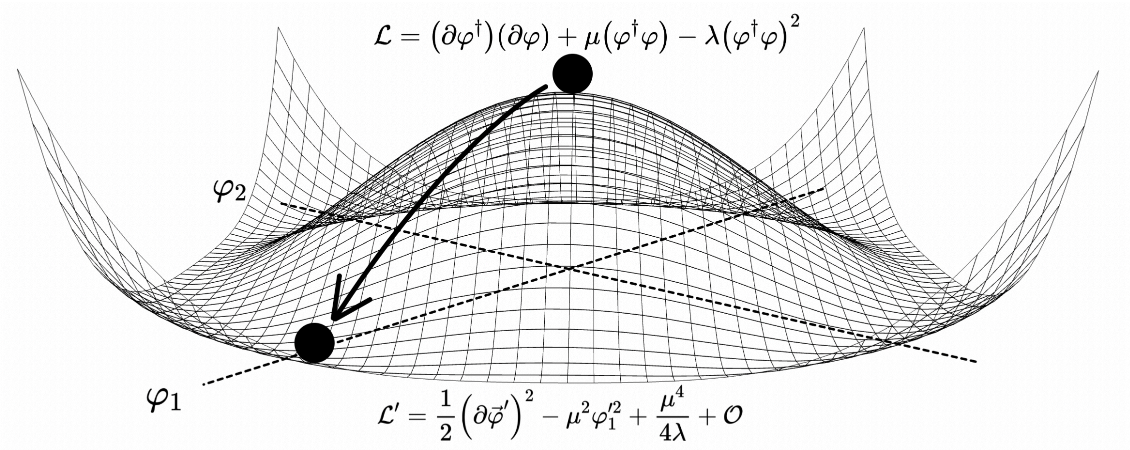


Figure 1: Appearance of spontaneous symmetry breaking

doviding into the real and imaginary parts

$$\begin{aligned}\mathrm{Re}\,\mathcal{L} &= (\partial\varphi^\dagger)(\partial\varphi) - \left(\sigma^2 - \tau^2\right)\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2 \\ \mathrm{Im}\,\mathcal{L} &= -2\sigma\tau\varphi^\dagger\varphi\end{aligned}$$

you will find when you look at the real part

$$\begin{aligned}\sigma^2 > \tau^2 &: \text{ massive} \\ \sigma^2 = \tau^2 &: \text{ massless} \\ \sigma^2 < \tau^2 &: \text{ SSB}\end{aligned}$$

what the imaginary part means?

## 1.2 Some examples of the diagram of the Lagrangian

For the Lagrangian

$$\mathcal{L} = \partial\varphi^\dagger\partial\varphi - m^2\varphi^\dagger\varphi - \lambda\left(\varphi^\dagger\varphi\right)^2$$

we separete the field into the real and imaginary $\varphi = \frac{1}{\sqrt{2}}\left(\varphi_1 + i\varphi_2\right)$

$$\mathcal{L} = \frac{1}{2}\left(\partial\varphi_1\right)^2 + \frac{1}{2}\left(\partial\varphi_2\right)^2 - \frac{m^2}{2}\varphi_1^2 - \frac{m^2}{2}\varphi_2^2 - \frac{\lambda}{4}\left(\varphi_1^2 + \varphi_2^2\right)^2$$

then the term $-\frac{\lambda}{2}\varphi_1^2\varphi_2^2$ provides the following diagram. Actually

$$Z = \int D\varphi_1 D\varphi_2 e^{i\int d^4x - \frac{1}{2}\varphi_1\left(\partial^2+m^2\right)\varphi_1 - \frac{1}{2}\varphi_2\left(\partial^2+m^2\right)\varphi_2 - \frac{\lambda}{4}\left(\varphi_1^2+\varphi_2^2\right)^2 + J_1\varphi_1 + J_2\varphi_2}$$

expanding

$$\begin{aligned}&= \left(\int D\varphi_1 e^{i\int d^4x - \frac{1}{2}\varphi_1\left(\partial^2+m^2\right)\varphi_1}\right)\cdot\left(\int D\varphi_2 e^{i\int d^4x - \frac{1}{2}\varphi_2\left(\partial^2+m^2\right)\varphi_2}\right) \\ &\times\left(1 - \frac{i\lambda}{4}\int dw\left(\varphi_1^2+\varphi_2^2\right)^2 + \cdots\right) \\ &\times\left(1 + i\int dx J_1\varphi_1 - \frac{1}{2}\int dx_1 dx_2 J_1(x_1)J_1(x_2)\varphi_1(x_1)\varphi_1(x_2) + \cdots\right) \\ &\times\left(1 + i\int dy J_2\varphi_2 - \frac{1}{2}\int dy_1 dy_2 J_2(y_1)J_2(y_2)\varphi_2(y_1)\varphi_2(y_2) + \cdots\right)\end{aligned}$$

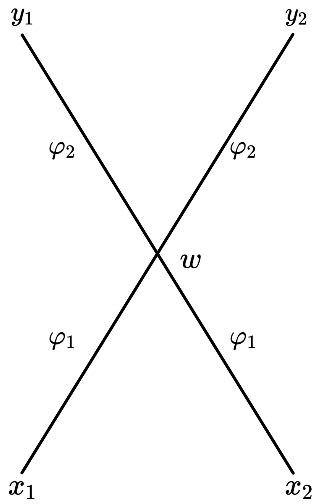


Figure 2: Diagram

taking the term corresponds to the diagram

$$\begin{aligned} I = &-\frac{i\lambda}{16}\int dx_1 dx_2 J\left(x_1\right) J\left(x_2\right)\int dy_1 dy_2 J\left(y_1\right) J\left(y_2\right) \\ &\times \int dw \int D\varphi_1 \varphi_1\left(x_1\right)\varphi_1\left(x_2\right)\varphi_1(w)^2 e^{i\int d^4x - \frac{1}{2}\varphi_1\left(\partial^2+m^2\right)\varphi_1} \\ &\times \int D\varphi_2 \varphi_2\left(y_1\right)\varphi_2\left(y_2\right)\varphi_2(w)^2 e^{i\int d^4x - \frac{1}{2}\varphi_2\left(\partial^2+m^2\right)\varphi_2} \end{aligned}$$

and the applying wick theorem

$$\begin{aligned} &= -\frac{i\lambda}{4}\int dx_1 dx_2 dy_1 dy_2 dw J\left(x_1\right) J\left(x_2\right) J\left(y_1\right) J\left(y_2\right) \\ &\times D\left(x_1 - w\right) D\left(x_2 - w\right) D\left(w - y_1\right) D\left(w - y_2\right) \end{aligned}$$

so the amplitude is

$$I = -\frac{i\lambda}{4}\int dw D\left(x_1 - w\right) D\left(x_2 - w\right) D\left(w - y_1\right) D\left(w - y_2\right)$$

## 1.3 What will happen if we choose the field shifted differently?

The stable domain is the circle which is

$$\varphi_1^2 + \varphi_2^2 = \frac{\mu^2}{\lambda}$$

Privaiously I took the fields as

$$\left(\varphi_1, \varphi_2\right) \to \left(\varphi_1' + \frac{\mu}{\sqrt{\lambda}}, \varphi_2'\right)$$

however it would be ok as long as they are shifted on the circle, more generaly

$$\left(\varphi_1, \varphi_2\right) \to \left(\varphi_1' + \frac{\mu}{\sqrt{\lambda}}\cos\theta, \varphi_2' + \frac{\mu}{\sqrt{\lambda}}\sin\theta\right)$$

the Lagrangian was

$$\mathcal{L} = \frac{1}{2}\left(\partial\varphi_1\right)^2 + \frac{1}{2}\left(\partial\varphi_2\right)^2 + \frac{1}{2}\mu^2\left(\varphi_1^2 + \varphi_2^2\right) - \frac{\lambda}{4}\left(\varphi_1^2 + \varphi_2^2\right)^2$$

and shifting as the above, then we have

$$\mathcal{L} = \frac{1}{2}\left(\partial\varphi_1\right)^2 + \frac{1}{2}\left(\partial\varphi_2\right)^2 - \varphi_1^2\mu^2\cos^2\theta - \varphi_2^2\mu^2\sin^2\theta + \frac{\mu^4}{4\lambda} + \mathcal{O}$$

This one is the more general shifted Lagrangian, and you can obtain the formar one just putting $\theta = 0$. This implies the Lagrangian has continuous symmetry.

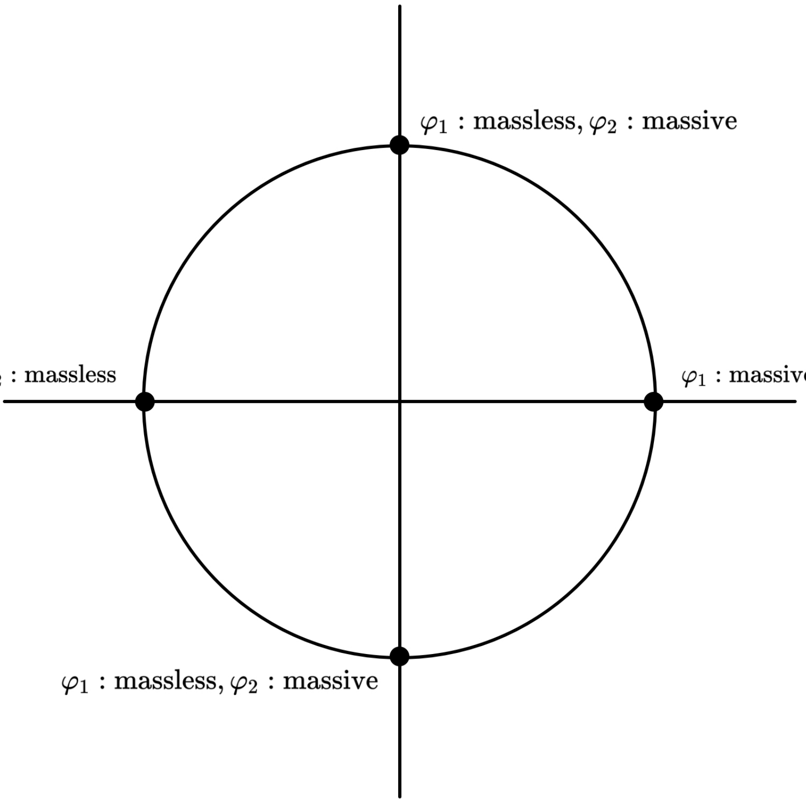


Figure 3: massive and massless points

## 1.4 Case.01

$$\mathcal{L} = \frac{1}{2}\left(\partial\varphi_1\right)^2 + \frac{1}{2}\left(\partial\varphi_2\right)^2 + \frac{\mu^2}{2}\varphi_1^2 + \frac{\mu^2}{2}\varphi_2^2 - \frac{\lambda}{4}\left(\varphi_1^2 + \varphi_2^2\right)^2 + \frac{\lambda}{2}\varphi_1^2\varphi_2^2$$

this Lagrangian's potential

$$V\left(\varphi_1, \varphi_2\right) = -\frac{\mu^2}{2}\left(\varphi_1^2 + \varphi_2^2\right) + \frac{\lambda}{4}\left(\varphi_1^4 + \varphi_2^4\right)$$

has the four minimas at

$$\left(\varphi_1, \varphi_2\right) = \left(\pm\frac{\mu}{\sqrt{\lambda}}, \pm\frac{\mu}{\sqrt{\lambda}}\right), \left(\pm\frac{\mu}{\sqrt{\lambda}}, \mp\frac{\mu}{\sqrt{\lambda}}\right)$$

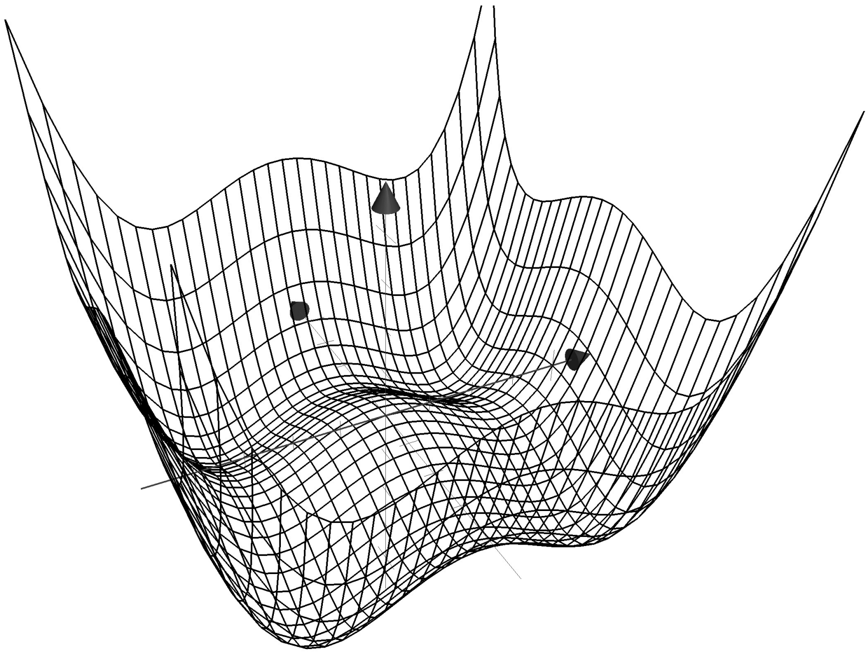

Figure 4: massive and massless points

Denote

$$\varphi_1 = \psi_1 + \frac{\mu}{\sqrt{\lambda}}, \varphi_2 = \psi_2 + \frac{\mu}{\sqrt{\lambda}}$$

then the potential will be

$$V(\psi_1, \psi_2) = \frac{1}{4}\lambda\psi_1^4 + \frac{1}{4}\lambda\psi_2^4 - \sqrt{\lambda}\mu\psi_1^3 - \sqrt{\lambda}\mu\psi_2^3 + \mu^2\psi_1^2 + \mu^2\psi_2^2 - \frac{\mu^4}{2\lambda}$$

so the new Lagrangian is as follows.

$$\mathcal{L} = \frac{\mu^4}{2\lambda} + \frac{1}{2}\left(\partial\psi_1\right)^2 + \frac{1}{2}\left(\partial\psi_2\right)^2 - \mu^2\left(\psi_1^2 + \psi_2^2\right) + \mathcal{O}$$

so you can find there is no massless field in this case, but $\psi_1, \psi_2$ are massive. This Lagrangian is not very interesting because there is no connection between $\psi_1, \psi_2$.

## 1.5 Case.02

Now consider the another case whose potential is as following

$$V\left(\varphi_1, \varphi_2\right) = -\frac{\mu^2}{2}\left(\varphi_1^2 + \varphi_2^2\right) + \frac{\lambda}{4}\left(\varphi_1^4 + \varphi_1^2\varphi_2^2 + \varphi_2^4\right)$$

the potential has minimas at

$$\begin{aligned}(\varphi_1, \varphi_2) &= \left(\pm\mu\sqrt{\frac{2}{3\lambda}}, \pm\mu\sqrt{\frac{2}{3\lambda}}\right)\\ &= \left(\pm\mu\sqrt{\frac{2}{3\lambda}}, \mp\mu\sqrt{\frac{2}{3\lambda}}\right)\end{aligned}$$

and shifting

$$\varphi_1 = \psi_1 + \mu\sqrt{\frac{2}{3\lambda}} \quad \varphi_2 = \psi_2 + \mu\sqrt{\frac{2}{3\lambda}}$$

the potential will be

$$V(\psi_1, \psi_2) = \sqrt{\frac{2}{3}}\sqrt{\lambda}\mu\psi_1^3 + \frac{1}{4}\lambda\psi_1^4 + \frac{1}{2}\sqrt{\frac{2}{3}}\sqrt{\lambda}\mu\psi_1^2\psi_2 + \frac{1}{2}\sqrt{\frac{2}{3}}\sqrt{\lambda}\mu\psi_1\psi_2^2 + \frac{1}{4}\lambda\psi_1^2\psi_2^2$$
$$+\sqrt{\frac{2}{3}}\sqrt{\lambda}\mu\psi_2^3 + \frac{1}{4}\lambda\psi_2^4 + \frac{2}{3}\mu^2\psi_1^2 + \frac{2}{3}\mu^2\psi_1\psi_2 + \frac{2}{3}\mu^2\psi_2^2 - \frac{\mu^4}{3\lambda}$$

so the shifted Lagrangian will be

$$\mathcal{L} = \frac{1}{2}\left(\partial\psi_1\right)^2 + \frac{1}{2}\left(\partial\psi_2\right)^2 - \frac{2}{3}\mu^2\left(\psi_1^2 + \psi_1\psi_2 + \psi_2^2\right) + \frac{\mu^4}{3\lambda} + \mathcal{O}$$

the both fields also massive fields.

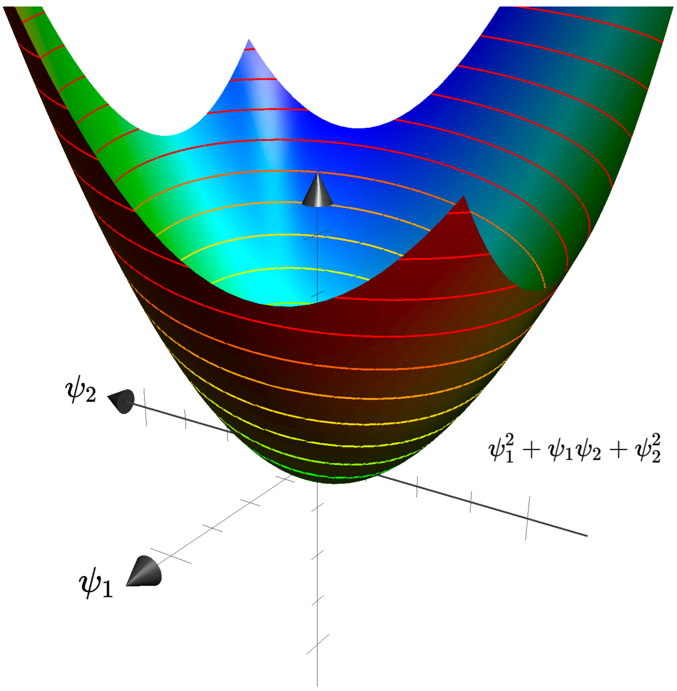


Figure 5: The figure of the shifted Lagrangian's potential

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Try to consider why the Lagrangian gets the massless field at the and with SSB.

$$\mathcal{L} = \left(\partial\varphi^\dagger\right)\left(\partial\varphi\right) + \mu\left(\varphi^\dagger\varphi\right) - \lambda\left(\varphi^\dagger\varphi\right)^2$$

One of the reason is the Lagrangian has spherical symmetry, but the last ones are not. And with SSB, the Lagrangian shows as the fact.

$$\mathcal{L}_{\mathrm{SSB}} = \frac{1}{2}\left(\partial\varphi_1\right)^2 + \frac{1}{2}\left(\partial\varphi_2\right)^2 - \varphi_1^2\mu^2\cos^2\theta - \varphi_2^2\mu^2\sin^2\theta + \frac{\mu^4}{4\lambda} + \mathcal{O}$$

Even if the Lagrangian changes from unstable to stable, a massless field does not necessarily appear. However, a massless field always appears for spherically symmetric Lagrangians.